

A Multigrid Method for Nonlinear Eigenvalue Problems: Version 2*

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Abstract

A multigrid method is proposed for solving nonlinear eigenvalue problems by the finite element method. With this new scheme, solving nonlinear eigenvalue problem is decomposed to a series of solutions of linear boundary value problems on multilevel finite element spaces and a series of small scale nonlinear eigenvalue problems. The computational work of this new scheme can reach almost the same as the solution of the corresponding linear boundary value problem. Therefore, this type of multilevel correction scheme improves the overfull efficiency of the nonlinear eigenvalue problem solving.

Keywords. nonlinear eigenvalue problem, finite element method, multilevel correction, multigrid.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

It is well known that solving large scale eigenvalue problems becomes a fundamental problem in modern science and engineering society. Among these eigenvalue problems, there exist many nonlinear eigenvalue problems [1, 2, 6, 7, 8, 9, 12, 15, 17, 20]. However, it is not an easy task to solve high-dimensional nonlinear eigenvalue problems which come from physical and chemical sciences.

The multigrid method and other efficient preconditioners provide an optimal order algorithm for solving boundary value problems since they can obtain the theoretical error by the linear scale computation work. We introduce the papers: Bramble and Zhang [4], Scott and Zhang [18], Xu [25], and books: Bramble [3], Brenner and Scott [5], Hackbusch [11], McCormick [16], Shaidurov [19] to the interested readers.

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Recently, we develop a type of multigrid method for linear eigenvalue problems [13, 14, 22, 23, 24]. Then the aim of this paper is to present a type of multigrid scheme for nonlinear eigenvalue problems based on the multilevel correction method [13]. With this method, solving nonlinear eigenvalue problem will not be more difficult than solving the corresponding linear boundary value problem. The multigrid method for nonlinear eigenvalue problem is based on a series of finite element spaces with different level of accuracy which can be built with the same way as the multilevel method for boundary value problems [25]. It is worth pointing out that besides the multigrid method, other types of numerical algorithms such as BPX multilevel preconditioners, algebraic multigrid method and domain decomposition preconditioners [5] can also act as the linear algebraic solvers for the multigrid method of the nonlinear eigenvalue problem.

The corresponding error and computational work estimates of the proposed multigrid scheme for the nonlinear eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal errors with an almost optimal computational work. The eigenvalue multigrid procedure can be described as follows: (1) solve the nonlinear eigenvalue problem in the coarsest finite element space; (2) solve an additional linear boundary value problem with multigrid method on the refined mesh using the previous obtained eigenvalue multiplying the corresponding eigenfunction as the load vector; (3) solve a nonlinear eigenvalue problem again on the finite element space which is constructed by combining the coarsest finite element space with the obtained eigenfunction approximation in step (2). Then go to step (2) for next loop until stop. In this method, we replace solving nonlinear eigenvalue problem on the finest finite element space by solving a series of linear boundary value problems with multigrid scheme in the corresponding series of finite element spaces and a series of nonlinear eigenvalue problems in the coarsest finite element space. So this multigrid method can improve the overfull efficiency of solving eigenvalue problems.

An outline of the paper goes as follows. In Section 2, we introduce finite element method for nonlinear eigenvalue problem and some assumptions in this paper. Two correction steps are given in Sections 3 and 4 based on fixed-point iteration and Newton iteration, respectively. In Section 5, we propose a type of multigrid algorithm for solving the nonlinear eigenvalue problem by finite element method. Section 6 is devoted to estimating the computational work for the multigrid method defined in Section 5. Some concluding remarks are given in the last section.

2 Finite element method for nonlinear eigenvalue problem

In this section, we introduce the finite element method for the nonlinear eigenvalue problem, some notation and error estimates of the finite element approximation for

eigenvalue problems. The letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols \lesssim , \gtrsim and \approx will be used in this paper. That $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes (see, e.g., [25]). We use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms, semi-norms [5, 10]. For $p = 2$, denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega}$ is understood in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$, and (\cdot, \cdot) is the standard $L^2(\Omega)$ inner product.

In this paper, we are concerned with the following nonlinear eigenvalue problem: Find (λ, u) such that

$$\begin{cases} -\Delta u + f(x, u) &= \lambda u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u^2 d\Omega &= 1, \end{cases} \quad (2.1)$$

where $\Omega \subset \mathcal{R}^d$ denotes the computing domain and $f(x, u)$ is a smooth enough function such that the eigenvalue problem (2.1) has only real eigenvalues.

In this paper, we set $V = H_0^1(\Omega)$. For the aim of finite element discretization, we define the corresponding weak eigenvalue problem as follows:

Find $(\lambda, u) \in \mathcal{R} \times V$ such that $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V, \quad (2.2)$$

where

$$a(u, v) := \int_{\Omega} (\nabla u \nabla v + f(x, u)v) d\Omega, \quad b(u, v) := \int_{\Omega} uv d\Omega.$$

Now, let us define the finite element approximations of the problem (2.2). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K . The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh \mathcal{T}_h , we can construct the linear finite element space denoted by $V_h \subset V$. In order to apply multigrid scheme, we start the process on the original mesh \mathcal{T}_H with the mesh size H and the original coarse linear finite element space V_H defined on the mesh \mathcal{T}_H . We assume that $V_h \subset V$ is a family of finite-dimensional spaces that satisfy the following assumption: For any $w \in V$

$$\lim_{h \rightarrow 0} \inf_{v \in V_h} \|w - v\|_1 = 0. \quad (2.3)$$

The standard finite element method is to solve the following eigenvalue problem: Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathcal{R} \times V_h$ such that $b(\bar{u}_h, \bar{u}_h) = 1$ and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in V_h. \quad (2.4)$$

Then we define

$$\delta_h(u) = \inf_{v_h \in V_h} \|u - v_h\|_1. \quad (2.5)$$

For generality, we only state the following assumptions about the error estimate for the eigenpair approximation $(\bar{\lambda}_h, \bar{u}_h)$ defined by (2.4) (see, e.g., [6, 9]).

Assumption A1: The eigenpair approximation $(\bar{\lambda}_h, \bar{u}_h)$ of (2.4) has the following error estimates

$$\|u - \bar{u}_h\|_1 \lesssim \delta_h(u), \quad (2.6)$$

$$|\lambda - \bar{\lambda}_h| + \|u - \bar{u}_h\|_0 \lesssim \eta_a(V_h) \|u - \bar{u}_h\|_1, \quad (2.7)$$

where $\eta_a(V_h)$ depends on the finite dimensional space V_h and has the following property

$$\lim_{h \rightarrow 0} \eta_a(V_h) = 0, \quad \eta_a(\tilde{V}_h) \leq \eta_a(V_h) \text{ if } V_h \subset \tilde{V}_h \subset V. \quad (2.8)$$

Assumption A2: Assume V^h is a subspace of V_h . Let us define the eigenpair approximation (λ^h, u^h) by solving the eigenvalue problem as follows:

Find $(\lambda^h, u^h) \in \mathcal{R} \times V^h$ such that $b(u^h, u^h) = \lambda^h$ and

$$a(u^h, v^h) = \lambda^h b(u^h, v^h), \quad \forall v^h \in V^h. \quad (2.9)$$

Then the following error estimates hold

$$\|\bar{u}_h - u^h\|_1 \lesssim \delta_h(\bar{u}_h), \quad (2.10)$$

$$|\bar{\lambda}_h - \lambda^h| + \|\bar{u}_h - u^h\|_0 \lesssim \eta_a(V^h) \|\bar{u}_h - u^h\|_1, \quad (2.11)$$

where

$$\delta_h(\bar{u}_h) := \inf_{v^h \in V^h} \|\bar{u}_h - v^h\|_1. \quad (2.12)$$

In order to design and analyze the multilevel correction method for the nonlinear eigenvalue problems, we also need the following assumptions for the nonlinear function $f(\cdot, \cdot) : \mathcal{R} \times V \rightarrow \mathcal{R}$.

Assumption B: The nonlinear function $f(x, \cdot)$ has the following estimate

$$|(f(x, w) - f(x, v), \psi)| \lesssim \|w - v\|_0 \|\psi\|_1, \quad \forall w \in V, \quad \forall v \in V, \quad \forall \psi \in V. \quad (2.13)$$

Assumption C: The nonlinear function $f(x, \cdot)$ has the following estimate

$$|(f(x, w) - f(x, v) - f_v(x, v)(w - v), \psi)| \lesssim \|w - v\|_0 \|\psi\|_1, \quad \forall w \in V, \quad \forall v \in V, \quad \forall \psi \in V. \quad (2.14)$$

For more discussions about the function $f(x, \cdot)$, please refer to [6, 7, 27] and the papers cited therein.

3 One correction step based on fixed-point iteration

In this section, we introduce a type of correction step based on the fixed-point iteration to improve the accuracy of the current eigenpair approximation. This correction step contains solving an auxiliary linear boundary value problem with multigrid method in the finer finite element space and a nonlinear eigenvalue problem on the coarsest finite element space.

Assume we have obtained an eigenpair approximation $(\lambda_{h_k}, u_{h_k}) \in \mathcal{R} \times V_{h_k}$. Now we introduce a type of correction step to improve the accuracy of the current eigenpair approximation (λ_{h_k}, u_{h_k}) . Let $V_{h_{k+1}} \subset V$ be a finer finite element space such that $V_{h_k} \subset V_{h_{k+1}}$. Based on this finer finite element space, we define the following correction step.

Algorithm 3.1. *One Correction Step based on Fixed-point Iteration*

1. Define the following auxiliary boundary value problem:

Find $\hat{u}_{h_{k+1}} \in V_{h_{k+1}}$ such that

$$(\nabla \hat{u}_{h_{k+1}}, \nabla v_{h_{k+1}}) = \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}) - (f(x, u_{h_k}), v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}. \quad (3.1)$$

Solve this equation with multigrid method to obtain an approximation $\tilde{u}_{h_{k+1}} \in V_{h_{k+1}}$ with error estimate

$$\|\hat{u}_{h_{k+1}} - \tilde{u}_{h_{k+1}}\|_a \leq C\eta_a(V_{h_k})\delta_{h_k}(u). \quad (3.2)$$

2. Define a new finite element space $V_{H,h_{k+1}} = V_H + \text{span}\{\tilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:

Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $b(u_{h_{k+1}}, u_{h_{k+1}}) = 1$ and

$$a(u_{h_{k+1}}, v_{H,h_{k+1}}) = \lambda_{h_{k+1}} b(u_{h_{k+1}}, v_{H,h_{k+1}}), \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}. \quad (3.3)$$

Summarize above two steps into

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = \text{Correction}(V_H, \lambda_{h_k}, u_{h_k}, V_{h_{k+1}}).$$

Theorem 3.1. Assume **Assumptions A1, A2** and **B** hold. The resultant approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by Algorithm 3.1 and the eigenpair approximation $(\bar{\lambda}_{h_{k+1}}, \bar{u}_{h_{k+1}})$ by the direct finite element method in $V_{h_{k+1}}$ have the following estimates

$$\|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \lesssim \varepsilon_{h_{k+1}}(u), \quad (3.4)$$

$$|\bar{\lambda}_{h_{k+1}} - \lambda_{h_{k+1}}| + \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_0 \lesssim \eta_a(V_H)\|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1, \quad (3.5)$$

$$|(f(x, \bar{u}_{h_{k+1}}) - f(x, u_{h_{k+1}}), v)| \lesssim \eta_a(V_H)\|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1\|v\|_1, \quad \forall v \in V. \quad (3.6)$$

where $\varepsilon_{h_{k+1}}(u) := \eta_a(V_{h_k})\delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{u}_{h_k} - \lambda_{h_k}|$.

Proof. From (2.4) and (3.1), the following inequalities hold for any $v_{h_{k+1}} \in V_{h_{k+1}}$

$$\begin{aligned}
& (\nabla(\bar{u}_{h_{k+1}} - \hat{u}_{h_{k+1}}), \nabla v_{h_{k+1}}) \\
&= b(\bar{\lambda}_{h_{k+1}} \bar{u}_{h_{k+1}} - \lambda_{h_k} u_{h_k}, v_{h_{k+1}}) + (f(x, u_{h_k}) - f(x, \bar{u}_{h_k}), v_{h_{k+1}}) \\
&\lesssim (|\bar{\lambda}_{h_{k+1}} - \lambda_{h_k}| + \|\bar{u}_{h_{k+1}} - u_{h_k}\|_0) \|v_{h_{k+1}}\|_1 \\
&\lesssim (|\bar{\lambda}_{h_{k+1}} - \bar{\lambda}_{h_k}| + |\bar{\lambda}_{h_k} - \lambda_{h_k}| + \|\bar{u}_{h_{k+1}} - \bar{u}_{h_k}\|_0 + \|\bar{u}_{h_k} - u_{h_k}\|_0) \|v_{h_{k+1}}\|_1 \\
&\lesssim (\eta_a(V_{h_k}) \delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{u}_{h_k} - \lambda_{h_k}|) \|v_{h_{k+1}}\|_1.
\end{aligned}$$

Then we have

$$\|\bar{u}_{h_{k+1}} - \hat{u}_{h_{k+1}}\|_1 \lesssim \eta_a(V_{h_k}) \delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{u}_{h_k} - \lambda_{h_k}|. \quad (3.7)$$

Combining (3.7) and the accuracy (3.2) leads to the following estimate

$$\|\bar{u}_{h_{k+1}} - \tilde{u}_{h_{k+1}}\|_1 \lesssim \eta_a(V_{h_k}) \delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{u}_{h_k} - \lambda_{h_k}|. \quad (3.8)$$

Now we come to estimate the error for the eigenpair solution $(\lambda_{h_{k+1}}, u_{h_{k+1}})$ of problem (3.3). Based on **Assumptions A1, A2** and **B**, and the definition of $V_{H,h_{k+1}}$, the following estimates hold

$$\|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \lesssim \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}} \|\bar{u}_{h_{k+1}} - v_{H,h_{k+1}}\|_1 \lesssim \|\bar{u}_{h_{k+1}} - \tilde{u}_{h_{k+1}}\|_1, \quad (3.9)$$

and

$$|\bar{\lambda}_{h_{k+1}} - \lambda_{h_{k+1}}| + \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_0 \lesssim \eta_a(V_{H,h_{k+1}}) \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1, \quad (3.10)$$

$$|(f(x, \bar{u}_{h_{k+1}}) - f(x, u_{h_{k+1}}), v)| \lesssim \eta_a(V_{H,h_{k+1}}) \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \|v\|_1, \quad \forall v \in V. \quad (3.11)$$

From (2.8), (3.8), (3.9), (3.10) and (3.11), we can obtain the desired results (3.4), (3.5) and (3.6). \square

4 One correction step based on Newton iteration

In this section, we present another type of correction step based on Newton iteration (always has better convergence property) to improve the accuracy of the given eigenpair approximations. This correction method also contains solving an auxiliary linear boundary value problem with multigrid method in the finer finite element space and a nonlinear eigenvalue problem on the coarsest finite element space.

Similarly, assume we have obtained an eigenpair approximation $(\lambda_{h_k}, u_{h_k}) \in \mathcal{R} \times V_{h_k}$. Let $V_{h_{k+1}} \subset V$ be a finer finite element space such that $V_{h_k} \subset V_{h_{k+1}}$.

In this section, we define the bilinear form $a_{h_k}(w, v)$ as follows

$$a_{h_k}(w, v) = (\nabla w, \nabla v) + (f_u(x, u_{h_k}) w, v). \quad (4.1)$$

Here, we assume the linearized operator $L_u := -\Delta + f_u(x, u)$ is nonsingular and u_{h_k} is close enough to u such that the following properties hold [27, Lemma 2.1]

$$\sup_{0 \neq v_{h_{k+1}} \in V_{h_{k+1}}} \frac{a_{h_k}(w_{h_{k+1}}, v_{h_{k+1}})}{\|v_{h_{k+1}}\|_1} \gtrsim \|w_{h_{k+1}}\|_1, \quad \forall w_{h_{k+1}} \in V_{h_{k+1}}, \quad (4.2)$$

$$|a_{h_k}(w, v)| \lesssim \|w\|_1 \|v\|_1, \quad \forall w \in V, \forall v \in V. \quad (4.3)$$

Now we define the correction step as follows.

Algorithm 4.1. *One Correction Step based on Newton Iteration*

1. Define the following auxiliary boundary value problem:

Find $\widehat{e}_{h_{k+1}} \in V_{h_{k+1}}$ such that

$$\begin{aligned} a_{h_k}(\widehat{e}_{h_{k+1}}, v_{h_{k+1}}) &= \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}) - (\nabla u_{h_k}, \nabla v_{h_{k+1}}), \\ &\quad - (f(x, u_{h_k}), v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}. \end{aligned} \quad (4.4)$$

Solve this equation with multigrid method [19, 26] to obtain an approximation $\widetilde{e}_{h_{k+1}} \in V_{h_{k+1}}$ with error estimate $\|\widehat{e}_{h_{k+1}} - \widetilde{e}_{h_{k+1}}\|_a \leq C\delta_{h_{k+1}}(u)$ and set $\widetilde{u}_{h_{k+1}} = u_{h_k} + \widetilde{e}_{h_{k+1}}$.

2. Define a new finite element space $V_{H, h_{k+1}} = V_H + \text{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:

Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H, h_{k+1}}$ such that $b(u_{h_{k+1}}, u_{h_{k+1}}) = 1$ and

$$a(u_{h_{k+1}}, v_{H, h_{k+1}}) = \lambda_{h_{k+1}} b(u_{h_{k+1}}, v_{H, h_{k+1}}), \quad \forall v_{H, h_{k+1}} \in V_{H, h_{k+1}}. \quad (4.5)$$

Summarize above two steps onto

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = \text{Correction}(V_H, \lambda_{h_k}, u_{h_k}, V_{h_{k+1}}).$$

Theorem 4.1. Assume Assumptions A1, A2 and C hold. The resultant approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by Algorithm 4.1 and the eigenpair approximation $(\bar{\lambda}_{h_{k+1}}, \bar{u}_{h_{k+1}})$ by the direct finite element method in $V_{h_{k+1}}$ have the following estimates

$$\|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \lesssim \varepsilon_{h_{k+1}}(u), \quad (4.6)$$

$$|\bar{\lambda}_{h_{k+1}} - \lambda_{h_{k+1}}| + \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_0 \lesssim \eta_a(V_H) \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1, \quad (4.7)$$

$$\begin{aligned} &| (f(x, \bar{u}_{h_{k+1}}) - f(x, u_{h_{k+1}}) - f_u(x, u_{h_{k+1}})(\bar{u}_{h_{k+1}} - u_{h_{k+1}}), v) | \\ &\lesssim \eta_a(V_H) \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \|v\|_1, \quad \forall v \in V, \end{aligned} \quad (4.8)$$

where $\varepsilon_{h_{k+1}}(u) := \eta_a(V_{h_k})\delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{\lambda}_{h_k} - \lambda_{h_k}|$.

Proof. From (2.4) and (4.4), the following estimates hold for any $v_{h_{k+1}} \in V_{h_{k+1}}$

$$\begin{aligned}
& a_{h_k}(\bar{u}_{h_{k+1}} - u_{h_k} - \hat{e}_{h_{k+1}}, v_{h_{k+1}}) \\
&= a_{h_k}(\bar{u}_{h_{k+1}} - u_{h_k}, v_{h_{k+1}}) - b(\lambda_{h_k} u_{h_k}, v_{h_{k+1}}) + (\nabla u_{h_k}, \nabla v_{h_{k+1}}) \\
&\quad + (f(x, u_{h_k}), v_{h_{k+1}}) \\
&= (\nabla \bar{u}_{h_{k+1}}, \nabla v_{h_{k+1}}) + (f_u(x, u_{h_k})(\bar{u}_{h_{k+1}} - u_{h_k}), v_{h_{k+1}}) - b(\lambda_{h_k} u_{h_k}, v_{h_{k+1}}) \\
&\quad + (f(x, u_{h_k}), v_{h_{k+1}}) \\
&= (f(x, u_{h_k}) - f(x, \bar{u}_{h_{k+1}}) + f_u(x, u_{h_k})(\bar{u}_{h_{k+1}} - u_{h_k}), v_{h_{k+1}}) \\
&\quad + b(\bar{\lambda}_{h_{k+1}} \bar{u}_{h_{k+1}} - \lambda_{h_k} u_{h_k}, v_{h_{k+1}}) \\
&\lesssim (\|\bar{u}_{h_{k+1}} - u_{h_k}\|_0 + |\bar{\lambda}_{h_{k+1}} - \lambda_{h_k}|) \|v_{h_{k+1}}\|_0 \\
&\lesssim (\|\bar{u}_{h_{k+1}} - \bar{u}_{h_k}\|_0 + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{\lambda}_{h_{k+1}} - \bar{\lambda}_{h_k}| + |\bar{\lambda}_{h_k} - \lambda_{h_k}|) \|v_{h_{k+1}}\|_1 \\
&\lesssim (\eta_a(V_{h_k})\delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{\lambda}_{h_k} - \lambda_{h_k}|) \|v_{h_{k+1}}\|_1. \tag{4.9}
\end{aligned}$$

Combing (4.2) and (4.9), we have the following estimates

$$\begin{aligned}
\|\bar{u}_{h_{k+1}} - u_{h_k} - \hat{e}_{h_{k+1}}\|_1 &\lesssim \sup_{0 \neq v_{h_{k+1}} \in V_{h_{k+1}}} \frac{a_{h_k}(\bar{u}_{h_{k+1}} - u_{h_k} - \hat{e}_{h_{k+1}}, v_{h_{k+1}})}{\|v_{h_{k+1}}\|_1} \\
&\lesssim \eta_a(V_{h_k})\delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{\lambda}_{h_k} - \lambda_{h_k}|. \tag{4.10}
\end{aligned}$$

Then from (4.10) and the accuracy $\|\hat{e}_{h_{k+1}} - \tilde{e}_{h_{k+1}}\|_1 \lesssim \eta_a(V_{h_k})\delta_{h_k}(u)$, the following inequality hold

$$\|\bar{u}_{h_{k+1}} - \tilde{u}_{h_{k+1}}\|_1 \lesssim \eta_a(V_{h_k})\delta_{h_k}(u) + \|\bar{u}_{h_k} - u_{h_k}\|_0 + |\bar{\lambda}_{h_k} - \lambda_{h_k}|. \tag{4.11}$$

Now we come to estimate the error for the eigenpair solution $(\lambda_{h_{k+1}}, u_{h_{k+1}})$ of problem (4.5). Based on **Assumptions A1, A2** and **C**, and the definition of $V_{H, h_{k+1}}$, the following estimates hold

$$\|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \lesssim \inf_{v_{H, h_{k+1}} \in V_{H, h_{k+1}}} \|\bar{u}_{h_{k+1}} - v_{H, h_{k+1}}\|_1 \lesssim \|\bar{u}_{h_{k+1}} - \tilde{u}_{h_{k+1}}\|_1, \tag{4.12}$$

and

$$|\bar{\lambda}_{h_{k+1}} - \lambda_{h_{k+1}}| + \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_0 \lesssim \eta_a(V_{H, h_{k+1}}) \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1, \tag{4.13}$$

$$\begin{aligned}
& |(f(x, \bar{u}_{h_{k+1}}) - f(x, u_{h_{k+1}}) - f_u(x, u_{h_{k+1}})(\bar{u}_{h_{k+1}} - u_{h_{k+1}}), v)| \lesssim \\
& \eta_a(V_{H, h_{k+1}}) \|\bar{u}_{h_{k+1}} - u_{h_{k+1}}\|_1 \|v\|_1, \quad \forall v \in V, \tag{4.14}
\end{aligned}$$

From (2.8), (4.11), (4.12), (4.13) and (4.14), the desired results (4.6), (4.7) and (4.8) can be obtained and the proof is complete. \square

5 Multigrid scheme for the eigenvalue problem

In this section, we introduce a type of multigrid correction scheme based on the *One Correction Step* defined in Algorithms 3.1 and 4.1. This type of multigrid method

can obtain the optimal error estimate as same as solving the nonlinear eigenvalue problem directly on the finest finite element space.

In order to do multigrid scheme, we define a sequence of triangulations \mathcal{T}_{h_k} of Ω determined as follows. Suppose \mathcal{T}_{h_1} is produced from \mathcal{T}_H by regular refinement and let \mathcal{T}_{h_k} be obtained from $\mathcal{T}_{h_{k-1}}$ via regular refinement (produce β^d subelements) such that

$$h_k \approx \frac{1}{\beta} h_{k-1}, \quad k = 2, \dots, n.$$

Based on this sequence of meshes, we construct the corresponding linear finite element spaces such that

$$V_H \subseteq V_{h_1} \subset V_{h_2} \subset \dots \subset V_{h_n}, \quad (5.1)$$

and the following relation of approximation errors hold

$$\delta_{h_k}(u) \approx \frac{1}{\beta} \delta_{h_{k-1}}(u), \quad k = 2, \dots, n. \quad (5.2)$$

Algorithm 5.1. *Eigenvalue Multigrid Scheme*

1. Construct a series of nested finite element spaces $V_{h_1}, V_{h_2}, \dots, V_{h_n}$ such that (5.1) and (5.2) hold.

2. Solve the following nonlinear eigenvalue problem:

Find $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$ such that $b(u_{h_1}, u_{h_1}) = 1$ and

$$a(u_{h_1}, v_{h_1}) = \lambda_{h_1} b(u_{h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}. \quad (5.3)$$

3. Do $k = 1, \dots, n - 1$

Obtain a new eigenpair approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by a correction step defined by Algorithm 3.1 or 4.1

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = \text{Correction}(V_H, \lambda_{h_k}, u_{h_k}, V_{h_{k+1}}). \quad (5.4)$$

End Do

Finally, we obtain an eigenpair approximation $(\lambda_{h_n}, u_{h_n}) \in \mathcal{R} \times V_{h_n}$.

Theorem 5.1. Assume we have conditions of Theorem 3.1 for Algorithm 5.1 with the correction step defined by Algorithm 3.1, or conditions of Theorem 4.1 for Algorithm 5.1 with the correction step defined by Algorithm 4.1. After implementing Algorithm 5.1, the resultant eigenpair approximation (λ_{h_n}, u_{h_n}) has the following error estimates

$$\|\bar{u}_{h_n} - u_{h_n}\|_1 \lesssim \beta^2 \eta_a(V_{h_n}) \delta_{h_n}(u), \quad (5.5)$$

$$|\bar{\lambda}_{h_n} - \lambda_{h_n}| + \|\bar{u}_{h_n} - u_{h_n}\|_0 \lesssim \eta_a(V_{h_n}) \delta_{h_n}(u). \quad (5.6)$$

under the condition $C\beta\eta_a^2(V_H) < 1$ for the constant C hidden in concerned inequalities.

Proof. Here we only give the proof for the case of the correction step defined by Algorithm 3.1 and the proof for Algorithm 4.1 case can be given similarly.

From the definition of Algorithm 5.1, we know that $\bar{u}_{h_1} = u_{h_1}$, $\bar{\lambda}_{h_1} = \lambda_{h_1}$. When $k = 2$, from Theorem 3.1 and Algorithm 5.1, the following estimates hold

$$\|\bar{u}_{h_2} - u_{h_2}\|_1 \lesssim \eta_a(V_{h_1})\delta_{h_1}(u), \quad (5.7)$$

$$\begin{aligned} |\bar{\lambda}_{h_2} - \lambda_{h_1}| + \|\bar{u}_{h_2} - u_{h_2}\|_0 &\lesssim \eta_a(V_H)\|\bar{u}_{h_2} - u_{h_2}\|_1 \\ &\leq \eta_a(V_H)\eta_a(V_{h_1})\delta_{h_1}(u), \end{aligned} \quad (5.8)$$

$$\begin{aligned} |(f(x, \bar{u}_{h_2}) - f(x, u_{h_2}), v)| &\lesssim \eta_a(V_H)\|\bar{u}_{h_2} - u_{h_2}\|_1\|v\|_1 \\ &\lesssim \eta_a(V_H)\eta_a(V_{h_1})\delta_{h_1}(u)\|v\|_1, \quad \forall v \in V. \end{aligned} \quad (5.9)$$

Based on Theorem 3.1, (5.2), (5.7)-(5.9) and recursive argument, the final eigenfunction approximation u_{h_n} has the following estimates

$$\begin{aligned} \|\bar{u}_{h_n} - u_{h_n}\|_1 &\lesssim \eta_a(V_{h_{n-1}})\delta_{h_{n-1}}(u) + \|\bar{u}_{h_{n-1}} - u_{h_{n-1}}\|_0 + |\bar{\lambda}_{h_{n-1}} - \lambda_{h_{n-1}}| \\ &\lesssim \eta_a(V_{h_{n-1}})\delta_{h_{n-1}}(u) + \eta_a(V_H)\|\bar{u}_{h_{n-1}} - u_{h_{n-1}}\|_1 \\ &\lesssim \eta_a(V_{h_{n-1}})\delta_{h_{n-1}}(u) + \eta_a(V_H)\eta_a(V_{h_{n-2}})\delta_{h_{n-2}}(u) \\ &\quad + \eta_a^2(V_H)\|\bar{u}_{h_{n-2}} - u_{h_{n-2}}\|_1 \\ &\lesssim \sum_{k=1}^{n-1} (\eta_a(V_H))^{n-k-1} \eta_a(V_{h_k})\delta_{h_k}(u) \\ &\lesssim \left(\sum_{k=1}^{n-1} (\beta^2 \eta_a(V_H))^{n-k-1} \right) \beta^2 \eta_a(V_{h_n})\delta_{h_n}(u) \\ &\lesssim \frac{1}{1 - \beta^2 \eta_a(V_H)} \beta^2 \eta_a(V_{h_n})\delta_{h_n}(u) \lesssim \beta^2 \eta_a(V_{h_n})\delta_{h_n}(u). \end{aligned} \quad (5.10)$$

This is the desired result (5.5). Similarly to the proof for Theorem 3.1, we can obtain the result (5.6) and the proof is complete. \square

Remark 5.1. *The results (5.5) and (5.6) mean that eigenpair approximation by the multigrid method have the same accuracy both in $L^2(\Omega)$ and $H^1(\Omega)$ as we solve the nonlinear eigenvalue problem directly by the finite element method.*

Corollary 5.1. *Under the conditions of Theorem 5.1, the eigenpair approximation (λ_{h_n}, u_{h_n}) by the multigrid method defined by Algorithm 5.1 has the following error estimates*

$$\|u - u_{h_n}\|_1 \lesssim \delta_{h_n}(u), \quad (5.11)$$

$$|\lambda - \lambda_{h_n}| + \|u - u_{h_n}\|_0 \lesssim \eta_a(V_{h_n})\delta_{h_n}(u). \quad (5.12)$$

6 Work estimate of eigenvalue multigrid scheme

In this section, we estimate the computational work for *Eigenvalue Multigrid Scheme* defined by Algorithm 5.1. We will show that Algorithm 5.1 makes solving eigenvalue

problem need almost the same work as solving the corresponding linear boundary value problem by the multigrid method.

First, we define the dimension of each level linear finite element space as

$$N_k := \dim V_{h_k}, \quad k = 1, \dots, n.$$

Then we have

$$N_k \approx \left(\frac{1}{\beta}\right)^{d(n-k)} N_n, \quad k = 1, \dots, n. \quad (6.1)$$

The computational work for the second step in Algorithm 3.1 or 4.1 is different from the linear eigenvalue problems [13, 22, 23, 24]. In this step, we need to solve a nonlinear eigenvalue problem (3.3) or (4.5). Always, some type of nonlinear iteration method (self-consistent iteration or Newton type iteration) is used to solve this nonlinear eigenvalue problem. In each nonlinear iteration step, we need to build the matrix on the finite element space V_{H,h_k} ($k = 2, \dots, n$) which needs the computational work $\mathcal{O}(N_k)$. Fortunately, the matrix building can be carried out by the parallel way easily in the finite element space since it has no data transfer.

Theorem 6.1. *Assume we use m computing-nodes in Algorithm 5.1, the nonlinear eigenvalue problem solved in the coarse spaces V_{H,h_k} ($k = 1, \dots, n$) and V_{h_1} need work $\mathcal{O}(M_H)$ and $\mathcal{O}(M_{h_1})$, respectively, and the work of multigrid method for solving the boundary value problem in V_{h_k} be $\mathcal{O}(N_k)$ for $k = 2, 3, \dots, n$. Let ϖ denote the nonlinear iteration times when we solve the nonlinear eigenvalue problem (3.3) or (4.5). Then in each computational node, the work involved in Algorithm 5.1 has the following estimate*

$$\text{Total work} = \mathcal{O}\left((1 + \frac{\varpi}{m})N_n + M_H \log N_n + M_{h_1}\right). \quad (6.2)$$

Proof. Let W_k denote the work in any processor of the correction step in the k -th finite element space V_{h_k} . Then with the correction definition, we have

$$W_k = \mathcal{O}\left(N_k + M_H + \varpi \frac{N_k}{m}\right). \quad (6.3)$$

Iterating (6.3) and using the fact (6.1), we obtain

$$\begin{aligned} \text{Total work} &= \sum_{k=1}^n W_k = \mathcal{O}\left(M_{h_1} + \sum_{k=2}^n \left(N_k + M_H + \varpi \frac{N_k}{m}\right)\right) \\ &= \mathcal{O}\left(\sum_{k=2}^n \left(1 + \frac{\varpi}{m}\right)N_k + (n-1)M_H + M_{h_1}\right) \\ &= \mathcal{O}\left(\sum_{k=2}^n \left(\frac{1}{\beta}\right)^{d(n-k)} \left(1 + \frac{\varpi}{m}\right)N_n + M_H \log N_n + M_{h_1}\right) \end{aligned}$$

$$= \mathcal{O}\left(\left(1 + \frac{\varpi}{m}\right)N_n + M_H \log N_n + M_{h_1}\right). \quad (6.4)$$

This is the desired result and we complete the proof. \square

Remark 6.1. *Since we have a good enough initial solution $\tilde{u}_{h_{k+1}}$ in the second step of Algorithm 3.1 or 4.1, then solving the nonlinear eigenvalue problem (3.3) or (4.5) always does not need many nonlinear iteration times (always $\varpi \leq 3$). In this case, the complexity in each computational node will be $\mathcal{O}(N_n)$ provided $M_H \ll N_n$ and $M_{h_1} \leq N_n$.*

7 Concluding remarks

In this paper, we give a type of multigrid scheme to solve nonlinear eigenvalue problems. The idea here is to use the multilevel correction method to transform the solution of the nonlinear eigenvalue problem to a series of solutions of the corresponding linear boundary value problems with multigrid method and a series of nonlinear eigenvalue problems on the coarsest finite element space. The proposed multigrid method can be applied to practical nonlinear eigenvalue problems [6, 7, 8, 9].

We can replace the multigrid method by other types of efficient iteration schemes such as algebraic multigrid method, the type of preconditioned schemes based on the subspace decomposition and subspace corrections (see, e.g., [5, 25]), and the domain decomposition method (see, e.g., [21]). Furthermore, the framework here can also be coupled with parallel method and the adaptive refinement technique. These will be investigated in our future work.

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